Homework 5
Geometry

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Proposition 0.1 (Exercise 8-1)**.** *Let M be a smooth manifold with or without boundary, and let $A \subset M$ be a closed subset. Let X be a smooth vector field along A. Given any $U \subset M$ such that $A \subset U$, there exists a smooth global vector field $\widetilde{X}$ on M such that $\widetilde{X}_A = X$ and* $\operatorname{supp} \widetilde{X} \subset U$.

Proof. Let $U \subset M$. For each $p \in A$, choose a neighborhood W_p of p and a smooth vector field $\widetilde{X}_p$ that agrees with X on $W_p \cap A$, that is, $\widetilde{X}_p(x) = X(x)$ for $x \in W_p \cap A$. If we replace W_p with $W_p \cap U$, we can assume that $W_p \subset U$. Then the collection

$$\{W_p : p \in A\} \cup \{M \setminus A\}$$

is an open cover of M. Let $\{\psi_p : p \in A\} \cup \{\psi_0\}$ be a smooth partition of unity subordinate to this cover, with $\operatorname{supp} \psi_p \subset W_p$ and $\operatorname{supp} \psi_0 \subset M \setminus A$.

For $p \in A$, we know that $\psi_p \widetilde{X}_p$ is smooth on W_p. If we extend it to all of M by setting its value to be zero on $M \setminus \operatorname{supp} \psi_p$, then this extension is smooth, because $\psi_p \widetilde{X}_p$ is zero on $W_p \setminus \operatorname{supp} \psi_p$. Then we define

$$\widetilde{X}(x) = \sum_{p \in A} \psi_p(x) \widetilde{X}_p(x)$$

By construction, the collection $\{\operatorname{supp} \psi_p\}$ is locally finite, so the above sum only has finitely many nonzero terms for any neighborhood in M. Thus it is a sum of finitely many smooth functions on such a neighborhood, and thus it is locally smooth near every $q \in M$. Thus it is smooth on all of M.

We check that that $\widetilde{X}$ is an extension of X. Let $x \in A$. Then $\psi_0(x) = 0$. We also have $\widetilde{X}_p(x) = X(x)$ whenever $\psi_p(x) \neq 0$, so

$$\widetilde{X}(x) = \sum_{p \in A} \psi_p(x) X(x) = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x) \right) X(x) = (0 + 1) X(x) = X(x)$$

(Note that $\sum_{p \in A} \psi_p((x) = 1$ by the properties of a partition of unity.) Thus $\widetilde{X}$ agrees with X on A, so it is an extension. All that remains to show is that $\operatorname{supp} \widetilde{X} \subset U$. Because the collection of $\operatorname{supp}\psi_p$ is locally finite, using Lemma 1.13(b) and the fact that $\overline{\operatorname{supp}\psi_p} = \operatorname{supp}\psi_p$ because $\operatorname{supp}\psi_p$ is closed, we have

$$\operatorname{supp}\widetilde{X} = \overline{\bigcup_{p\in A}\operatorname{supp}\psi_p} = \bigcup_{p\in A}\overline{\operatorname{supp}\psi_p} = \bigcup_{p\in A}\operatorname{supp}\psi_p \subset U$$

Thus $\operatorname{supp}\widetilde{X} \subset U$ as needed. □

(Exercise 8-11a)
Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ be a vector field. Then in polar coordinates we have $x = r\cos\theta$ and $y \in r\sin\theta$, so

$$\begin{aligned}\frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - r\sin\theta\frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + r\cos\theta\frac{\partial}{\partial\theta}\end{aligned}$$

Then we can compute

$$\begin{aligned}X &= r\cos\theta\left(\cos\theta\frac{\partial}{\partial r} - r\sin\theta\frac{\partial}{\partial\theta}\right) + r\sin\theta\left(\sin\theta\frac{\partial}{\partial r} + r\cos\theta\frac{\partial}{\partial\theta}\right)\\ &= r\cos^2\theta\frac{\partial}{\partial r} - r^2\cos\theta\sin\theta\frac{\partial}{\partial\theta} + r\sin^2\theta\frac{\partial}{\partial r} + r^2\sin\theta\cos\theta\frac{\partial}{\partial\theta}\\ &= r(\sin^2\theta + \cos^2\theta)\frac{\partial}{\partial r}\\ &= r\frac{\partial}{\partial r}\end{aligned}$$

(Exercise 8-16b)
Let $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, Y = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$. We compute the Lie bracket:

$$\begin{aligned}[X,Y] &= XY - YX\\ &= \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) - \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\\ &= x\frac{\partial}{\partial y}y\frac{\partial}{\partial z} - x\frac{\partial}{\partial y}z\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}y\frac{\partial}{\partial z} + y\frac{\partial}{\partial x}z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}yddx + z\frac{\partial}{\partial y}x\frac{\partial}{\partial y} - z\frac{\partial}{\partial y}y\frac{\partial}{\partial x}\\ &= x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\end{aligned}$$